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BIFURCATION OF COMMON LEVELS OF FIRST INTEGRALS OF THE KOVALEVSKAYA PROBLEM*

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The structure of integral manifolds in the Kovalevskaya problem of a heavy solid about a fixed point is considered. An analytic definition of a bifurcation set is obtained, and bifurcation diagrams are constructed. The number of two-dimensional toruses that appear in the composition of the integral manifold is indicated for each connected component, additional to the bifurcation set in the space of first integral constants.

The solution of the problem of the motion of a solid about a fixed point, as formulated by Kovalevskaya /1/, has been dealt with in many publications. We shall mention only a few of them. Appel'rot was the first to identify four classes of motion of the Kovalevskaya gyroscope /2/. A more detailed study of particular motions appeared in /3/, where a geometric treatment of Appel'rot classes is presented as corresponding to parts of the surface of multiple roots of the Kovalevskaya polynomial in the space of first-integral constants. The hodograph was used in /4, 5/ for a complete study of the motion belonging to the first and second classes, and the so-called particularly unusual motion of the third class in which the moving hodograph of the angular velocity of the body is a closed curve.

The set of zero measure corresponds to Appel'rot classes in the space of first integral constants. The remaining classes were not studied to any great extent, and it is only recently that their important qualitative properties were established /6/. It was assumed that the first Euler-Poisson equations are independent of the motions considered. However, it is still not known exactly at what values of the constant integrals the latter are independent. It is proved below that the Appel'rot classes correspond to the cases of integral dependence. The study of this question enables us to indicate in all cases the number of connected components of the integral manifold, each of which in the space of Euler-Poisson variables is a two-dimensional torus that carries conditionally periodic motions /7, 8/. The fact that integral manifolds, that do not degenerate when the Poincare parameter approaches zero, consist of two toruses is pointed out in /6/.

The investigation of integral manifolds as part of the solution of the problem of the topological analysis of classical dynamic systems can be traced back to Poincare and Birkhoff. It was formulated in modern terms by Smail /9/.

Finally, we note /10/, where, with some inaccuracies, eliminated in /11/ when investigating general cases, the particular problem of the bifurcation of the integrals of energy and areas is solved. The Kovalevskaya integral, and hence the complete integrability of the system, were ignored.

1. Let p, q, r be the components of the angular velocity vector $\boldsymbol{\omega}$, and ν_1, ν_2, ν_3 the components of the unit vector $\boldsymbol{\nu}$ of the vertical in the trihedron accompanying the solid. By a suitable selection of the moving axes and units of measurement, we reduce the Euler-Poisson equations in the Kovalevskaya problem to the form

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$$2p^{\cdot} = qr, \quad 2q^{\cdot} = -(pr + \nu_3), \quad r^{\cdot} = \nu_2 \tag{1.1}$$
$$\nu_1^{\cdot} = r\nu_2 - q\nu_3, \quad \nu_2^{\cdot} = p\nu_3 - r\nu_1, \quad \nu_3^{\cdot} = q\nu_1 - p\nu_2$$

Their first integrals are

$$2(p^2 + q^2) + r^2 - 2\nu_1 = 2h \tag{1.2}$$
$$2(p\nu_1 + q\nu_2) + r\nu_3 = 2l \tag{1.3}$$
$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1 \tag{1.4}$$
$$(p^2 - q^2 + \nu_1)^2 + (2pq + \nu_2)^2 = k \tag{1.5}$$

Let us recall the essence of Appel'rot classification. We introduce, as in /1/, the variables s_1 and s_2

$$s_{1,2} = h + [R(x_1, x_2) \mp \sqrt{R(x_1)R(x_2)}]/(x_1 - x_2)^2 \tag{1.6}$$

where

$$x_1 = p + iq, \quad x_2 = p - iq \tag{1.7}$$
$$R(x_1, x_2) = -x_1^2x_2^2 + 2hx_1x_2 + 2l(x_1 + x_2) + 1 - k \tag{1.8}$$
$$R(x) = -x^4 + 2hx^2 + 4lx + 1 - k \tag{1.9}$$

The dependence of the variables (1.6) on time is determined by the equations

$$(s_1 - s_2)^2 s_1^{\cdot 2} = -2\Phi(s_1), \quad (s_1 - s_2)^2 s_2^{\cdot 2} = -2\Phi(s_2).$$
$$\Phi(s) = (s - h + \sqrt{k})(s - h - \sqrt{k})\varphi(s)$$
$$\varphi(s) = s^3 - 2hs^2 + (h^2 + 1 - k)s - 2l^2$$

where $\varphi(s)$ is the resolvent of the Euler polynomial (1.9).

Motions that correspond to those constants of the first integrals for which the polynomial $\Phi(s)$ has a multiple root, were called the simplest by Appel'rot. The surface of multiple roots in $\mathbf{R}^3(h, k, l)$ consists of the plane

$$k = 0 \tag{1.10}$$

(the first class of simplest motions or the Delaunay class), of the surface

$$k = (h - 2l^2)^2 \tag{1.11}$$

(the second and third classes of simplest motions), and of the surface of multiple roots of the resolvent $\varphi(s)$, having the equation /3/

$$(1 - k)(h^2 + 1 - k)^2 - 2[9h(1 - k) + h^3]l^2 + 27l^4 = 0 \tag{1.12}$$

(the fourth class of simplest motions). Surface (1.12) may be represented in parametric form

$$h = s + \frac{l^2}{s^2}, \quad k = 1 - \frac{2l^2}{s} + \frac{l^4}{s^4} \tag{1.13}$$

or

$$h = (x^3 - l)/x, \quad k = 1 + 2lx + x^4 \tag{1.14}$$

In (1.13) the parameter s represents the multiple root of $\varphi(s)$ and in (1.14) x is the multiple root of the original polynomial (1.9).

2. Let us revert to system (1.2)-(1.5). For fixed k, h, l it defines in $\mathbf{R}^6(\omega, \nu)$ the surface $J_{k,h,l}$ that is invariant to the phase stress (1.1), and called the integral manifold. Let us enumerate certain known facts that apply to this problem.

The point $(\omega, \nu) \in \mathbf{R}^6$ is called critical, if at that point the rank of Jacobi's matrix is less than four, i.e.

$$\operatorname{rank}\begin{Vmatrix} 2p & 2q & -1 & 0 & r & 0 \\ 2\nu_1 & 2\nu_2 & 2p & 2q & \nu_3 & r \\ 0 & 0 & \nu_1 & \nu_2 & 0 & \nu_3 \\ 2(p\eta_1 + q\eta_2) & 2(p\eta_2 - q\eta_1) & \eta_1 & \eta_2 & 0 & 0 \end{Vmatrix} < 4 \tag{2.1}$$

$$\eta_1 = p^2 - q^2 + \nu_1, \quad \eta_2 = 2pq + \nu_2 \tag{2.2}$$

The set $(k, h, l) \in \mathbf{R}^3$ is called a regular value, if on the respective surface $J_{k,h,l}$ there are no critical points (particularly if $J_{k,h,l} = \varnothing$). The set (k, h, l) which is not a regular value, is, by definition, a critical value. Critical values fill the set Σ in $\mathbf{R}^3$, which is called bifurcational.

Because of the obvious compactness of the sets $J_{k,h,l}$ (which is already ensured by (1.2) and (1.4)) their differentiable type can only change on passing through Σ. By the Liouville-Arnold theorem for $(k, h, l) \in \mathbf{R}^3 \setminus \Sigma$ the set $J_{k,h,l}$ is the union of a finite number of two-dimensional toruses. For a complete investigation of integral manifolds it is therefore sufficient to construct the set Σ and establish the number of toruses in each of the connected components $\mathbf{R}^3 \setminus \Sigma$.

Let us consider some special cases. In (1.2)–(1.5) let

$$q = 0 \tag{2.3}$$

We introduce the values of h, l, k from (1.2), (1.3) and (1.5) into the polynomial (1.9) and its derivative, assuming in passing that $x = p$. We obtain

$$R(p) = (pr + \nu_3)^2, \quad R'(p) = 2r(pr + \nu_3) \tag{2.4}$$

We substitute into the matrix (2.1) the combination of rows with coefficients $p^2, p, 1, -1$ for the last row, respectively. After some elementary transformations which amount to cancelling common factors and transposing rows and columns, we obtain the condition

$$\operatorname{rank}\begin{Vmatrix} pr+\nu_3 & 0 & 0 & 0 & 0 & p(pr+\nu_3) \\ 0 & -1 & 0 & 0 & p & r \\ r & 2p & \nu_2 & 0 & \nu_1 & \nu_3 \\ \nu_3 & \nu_1 & 0 & \nu_2 & 0 & 0 \end{Vmatrix} < 4 \tag{2.5}$$

By equating to zero the determinant made up of the first columns we obtain $(pr + \nu_3)\nu_2 = 0$. If $pr + \nu_3 = 0$, then by virtue of (2.4) the polynomial (1.9) has a multiple root, and condition (1.14) with $x = p$ is satisfied. Let

$$\nu_2 = 0, \quad pr + \nu_3 \neq 0 \tag{2.6}$$

Condition (2.5) now takes the form $(p^2 + \nu_1)(2p\nu_3 - r\nu_1) = 0$. The case when $p^2 + \nu_1 = 0$ together with (2.3) and (2.6) yields (1.10). If, however,

$$2p\nu_3 - r\nu_1 = 0 \tag{2.7}$$

then from (1.4), (2.3) and (2.6) we obtain $h - 2l^2 = -(p^2 + \nu_1)$, $k = (p^2 + \nu_1)^2$, from which (1.11) follows. Thus all critical values attainable in the case of (2.3) correspond to Appel'rot classes.

Let us investigate one more possibility:

$$r = 0, \quad \nu_3 = 0, \quad q \neq 0 \tag{2.8}$$

In matrix (2.1) the last two columns are zeros. Equating to zero the remaining fourth-order determinant, we obtain $2(p^2 - q^2 + \nu_1)\nu_1\nu_2 - (2pq + \nu_2)(\nu_1^2 - \nu_2^2) = 0$, which enables us to introduce the undetermined multiplier $\varkappa$

$$p^2 - q^2 + \nu_1 = \varkappa(\nu_1^2 - \nu_2^2), \quad 2pq + \nu_2 = 2\varkappa\nu_1\nu_2 \tag{2.9}$$

Substituting these expressions into (1.5) and using (1.4) we obtain

$$\varkappa = \pm\sqrt{k} \tag{2.10}$$

From (1.4) and (2.9) we have

$$(p^2 + q^2)^2 = (\varkappa\nu_1 - 1)^2 + \varkappa^2\nu_2^2 \tag{2.11}$$

We will rewrite system (2.9) in the form

$$(\varkappa\nu_1 - 1)(p\nu_1 + q\nu_2) + \varkappa\nu_2(q\nu_1 - p\nu_2) = p(p^2 + q^2)$$
$$\varkappa\nu_2(p\nu_1 + q\nu_2) - (\varkappa\nu_1 - 1)(q\nu_1 - p\nu_2) = q(p^2 + q^2)$$

from which, taking into account (2.11), we obtain

$$\nu_1 = \frac{p^2 - q^2 + \varkappa}{\varkappa^2 - (p^2 + q^2)^2}, \quad \nu_2 = \frac{2pq}{\varkappa^2 - (p^2 + q^2)^2}$$

We subsitute the above expressions together with (2.8) into (1.2)–(1.4) and obtain

$$h = p^2 + q^2 - \frac{p^2 - q^2 + \varkappa}{\varkappa^2 - (p^2 + q^2)^2}, \quad l = \frac{p}{\varkappa - p^2 - q^2} \tag{2.12}$$

$$(p^2 + q^2)^4 - (1 + 2\varkappa^2)(p^2 + q^2)^2 - 2\varkappa(p^2 - q^2) - \varkappa^2(1 - \varkappa^2) = 0$$

from which $h - 2l^2 + \varkappa = 0$, which by virtue of (2.10) reduces to (1.11). Thus, under condition (2.8) also the critical values correspond to Appel'rot classes.

We shall show that on the assumption that

$$r^2 + \nu_3^2 \neq 0, \quad q \neq 0 \tag{2.13}$$

the system (1.2)−(1.5) has no new critical values.

At the beginning we note the identity $R(x_1, x_2) = (pr + \nu_3)^2 + q^2r^2$ which follows from (1.2)−(1.5), (1.7) and (1.8) so that under condition (2.13)

$$R(x_1, x_2) \neq 0, \quad x_1 \neq x_2 \tag{2.14}$$

We will introduce in addition to (1.7) the variables $\xi_1 = \eta_1 + i\eta_2$, $\xi_2 = \eta_1 - i\eta_2$ /1/. By (2.2) they are connected by a non-degenerate transformation with ν_1, ν_2. Eliminating from (1.2)−(1.5) the quantities r, ν_3, we obtain two relations /1/

$$\begin{gathered}\xi_1\xi_2 = k\\ R(x_2)\,\xi_1 + R(x_1)\,\xi_2 + R_1(x_1, x_2) + (x_1 - x_2)^2 k = 0\end{gathered} \tag{2.15}$$

where

$$R_1(x_1, x_2) = -2hx_1^2x_2^2 - 4l(x_1 + x_2)\,x_1x_2 - (1 - k)(x_1 + x_2)^2 + 2(1 - k)\,h - 4l^2 \tag{2.16}$$

The first inequality is sufficient for condition (2.1) to be equivalent to system (2.15) which determines the critical points

$$R(x_1)\,\xi_2 = R(x_2)\,\xi_1 \tag{2.17}$$

$$\begin{gathered}R'(x_1)\,\xi_2 + \partial R_1(x_1, x_2)/\partial x_1 + 2(x_1 - x_2)\,k = 0\\ R'(x_2)\,\xi_1 + \partial R_1(x_1, x_2)/\partial x_2 - 2(x_1 - x_2)\,k = 0\end{gathered} \tag{2.18}$$

Following /1/, we express ξ_1, ξ_2 from (2.15)

$$\begin{gathered}2R(x_2)\,\xi_1 = -[R_1(x_1, x_2) + (x_1 - x_2)^2 k] + W(x_1, x_2)\\ 2R(x_1)\,\xi_2 = -[R_1(x_1, x_2) + (x_1 - x_2)^2 k] - W(x_1, x_2)\\ (W(x_1, x_2) = \{[R_1(x_1, x_2) - (x_1 - x_2)^2 k]^2 - 4kR^2(x_1, x_2)\}^{1/2})\end{gathered} \tag{2.19}$$

Condition (2.17), thus, provides $W(x_1, x_2) = 0$. Using the notation of (2.10) we obtain

$$R_1(x_1, x_2) - (x_1 - x_2)^2 \varkappa^2 = 2\varkappa R(x_1, x_2) \tag{2.20}$$

Moreover, substituting ξ_1, ξ_2 from (2.19) into (2.18) we obtain

$$\begin{gathered}2(1 - \varkappa^2)^2 + 2(1 - \varkappa^2)[h^2 - 1 + x_1^2x_2^2 + 3l(x_1 + x_2)] +\\ 2h^2x_1^2x_2^2 + h[-(x_1 + x_2)^2 + 4l(x_1 + x_2)\,x_1x_2 - 4l^2] +\\ (x_1^2 + x_2^2)\,x_1x_2 + 2l(x_1^2x_2^2 - 2)(x_1 + x_2) + 4l^2(x_1^2 +\\ 3x_1x_2 + x_2^2) = 0\end{gathered} \tag{2.21}$$

$$[R_1(x_1, x_2) - (x_1 - x_2)^2 \varkappa^2]\,l + (x_1 + x_2 + 2lx_1x_2)\,R(x_1, x_2) = 0 \tag{2.22}$$

From (2.20) and (2.22) we have $[x_1 + x_2 + 2l(x_1x_2 + \varkappa)]\,R(x_1, x_2) = 0$, so that by assumption (2.14)

$$x_1 + x_2 = -2l(x_1x_2 + \varkappa) \tag{2.23}$$

We will substitute the expression obtained for $x_1 + x_2$ into (2.21) and (2.22). This gives

$$\begin{gathered}(\varkappa - h + 2l^2)[4l^2(x_1x_2 + \varkappa)^2 - (h + \varkappa)(x_1^2x_2^2 + 1 - \varkappa^2)] = 0\\ (\varkappa - h + 2l^2)[(x_1x_2 + \varkappa)^2 - 1] = 0\end{gathered}$$

Hence either $\varkappa - h + 2l^2 = 0$, which again leads to (1.13), or $x_1x_2 = \pm 1 - \varkappa$, $x_1 + x_2 = \mp 2l$, $2l^2 = (h + \varkappa)(1 \mp \varkappa)$. In the latter case a direct check gives $R(x_1, x_2) = 0$, but this contradicts (2.14). This proves that the bifurcational set Σ is part of the surface of multiple roots of the polynomial $\Phi(s)$, that corresponds to real solutions of (1.2)−(1.5).

Note that the set of critical points consists of those trajectories of (1.1) to which in Appel'rot's terminology the particularly unusual motions correspond (one of the quantities (1.6) remains constant during the motion).

3. The form of (1.11) and (1.14) implies that it is convenient to consider the cross section $\Sigma_l \subset \mathbf{R}^2(k, h)$ of the set Σ by planes $l = \mathrm{const}$. That method is more economical and suits the aim of the present investigation better than the method used in /3/, where, for instance, the solutions of (1.12) are studied relative to l and the projections of multiple

roots on the planes lh and lk. Since $\Sigma_l = \Sigma_{-l}$, we will limit our investigations to the case of $l \geqslant 0$.

Equation (1.11) defines in the kh plane a parabola whose vertex is at $(0, 2l^2)$.

Let us now consider the curve (1.12). When $l = 0$ it decomposes into the straight line $k = 1$ and the parabola $k = 1 + h^2$. If $l > 0$ we use (1.14). The curve investigated has a cupsidal point $x = (-l/2)^{1/3}$ and a vertical asymptote $k = 1$ $(x \to \pm 0, h \to \mp\infty)$. As $x \to \pm\infty$ both coordinates k and h approach $+\infty$ so that $(k(x), h(x))$ asymptotically approaches the respective curves $k = h^2 \pm 4l\sqrt{h} + 1$.

Generally, curve (1.14) has two points of intersection with parabola (1.11)

$$(k, h) = ((l^2 + 1)^2, l^2 - 1), \quad x = l \tag{3.1}$$

$$(k, h) = ((l^2 - 1)^2, l^2 + 1), \quad x = -l \tag{3.2}$$

and a point of tangency

$$(k, h) = \left(\frac{1}{16l^4}, \frac{1}{4l^2} + 2l^2\right), \quad x = -\frac{1}{2l} \tag{3.3}$$

The case when $l^2 = 1/2$, when points (3.2) and (3.3) coincide with the cupsidal point is an exception. When $l^2 = 4/(3\sqrt{3})$ the cupsidal point is on the axis $k = 0$ (which is obviously a part of the boundary of the region where motion occurs). One more singularity arises when $l = 1$: point (3.1) then passes from one branch of the parabola (1.11) to the other.

It is interesting to follow the transformation of curve (1.14) as $l \to 0$. The branch $-\infty < x < 0$ converts to the ray $\{k = 1, h \geqslant 0\}$ and the upper part of parabola $k = h^2 + 1$ $(h \geqslant 0)$ Consequently, the cupsidal point reaches point (1, 0). The branch $0 < x < +\infty$ joins with the ray $\{k = 1, h \leqslant 0\}$ and the same upper part of the parabola $k = h^2 + 1$. The tangency point of curve (1.14) and parabola (1.11) moves to infinity as $l \to 0$.

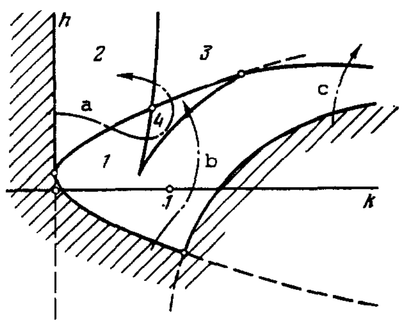

Fig.1

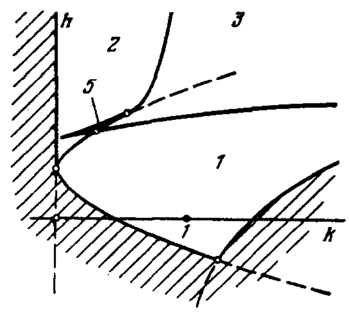

Fig.2

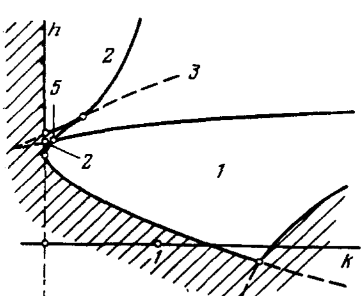

Fig.3

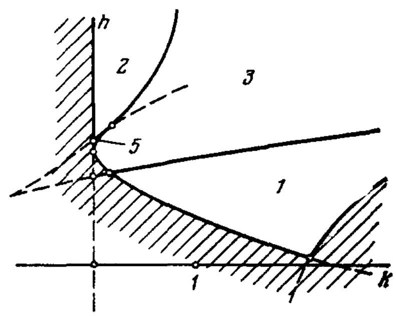

Fig.4

The form of the sets Σ_l is shown in Figs.1–4 for the following values of the area constants: $0 < l^2 < 1/2$; $1/2 < l^2 < 4/(3\sqrt{3})$; $4/(3\sqrt{3}) < l^2 < 1$; $l^2 > 1$.

4. As already noted, the form of the manifold $J_{k,h,l}$ can change only when point (k, h, l) passes through the critical value, i.e. through the set (1.10)–(1.12), as proved. Consequently in the shaded regions of the kh plane containing points with $k < 0$ or $h < -1$, $J_{k,h,l} = \varnothing$.

We will show that part of the upper branch

$$h = 2l^2 + \sqrt{k} \tag{4.1}$$

of the parabola (1.11) lying to the right of the point (3.3) of tangency to the curve (1.14) is not included in Σ_l. For this it is sufficient to show that under condition (4.1) the values

$$\sqrt{k} \leqslant 1/(4l^2) \tag{4.2}$$

correspond to real critical points (this corresponds to the statement /2, 3/ that the third class does not contain particularly noticeable motions when $\sqrt{k} > 1/(4l^2)$).

The values defined in (4.1) are reached, first of all, if (2.3), (2.6) and (2.7) are satisfied

$$2p\nu_3 - r\nu_1 = 0, \quad q = 0, \quad \nu_2 = 0 \tag{4.3}$$

secondly, under conditions (2.12) with $\varkappa = -\sqrt{k}$, and thirdly, in the case of (2.23) with $\varkappa = \sqrt{k}$.

The last two possibilities lead to the equation

$$l(p^2 + q^2) + p + l\sqrt{k} = 0 \tag{4.4}$$

which has real solutions for p and q if and only if the trinomial $lp^2 + p + l\sqrt{k}$ has real roots, which leads to the inequality (4.2).

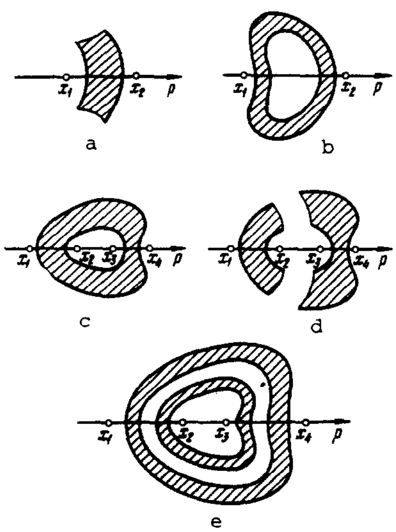

Fig.5

Let us consider once again case (4.3). From (1.3) and the first of (4.3) we have $\nu_1 = 4lp/(4p^2 + r^2)$, $\nu_3 = 2lr/(4p^2 + r^2)$. Substitution into (1.4) gives $4p^2 + r^2 = 4l^2$, after which formula (1.5) takes the form (4.4).

In Fig.1–4 the regions whose amalgamation in $\mathbf{R}^3(k, h, l)$ yields a connected component $\mathbf{R}^3 \setminus \Sigma$ are denoted by the same numbers. There are thus five components in which the integral manifolds are non-empty. To establish the number of toruses appearing in $J_{k,h,l}$ we shall consider the image of the latter in the plane pq, called in /2/ the region of real motions. It is not difficult to establish the connection between the subregions 1–5 in $\mathbf{R}^3 \setminus \Sigma$ and the cases considered in /2/. As the result, we obtain for subregions 1–5 the projections on the pq plane shown in Fig.5,a-e (the q axis is vertical, and x_i are the real roots of the polynomial $R(x)$).

Let us establish the number of prototypes of each inner point of the shaded sets. Since all of them reach the p axis, it is sufficient to do this for $q=0$. From (1.2)–(1.5), using the notation (1.8) and (2.16), we have

$$\nu_1 = p^2 + \frac{1}{2}r^2 - h, \quad r\nu_3 = -pr^2 + 2(l + hp - p^3) \tag{4.5}$$

$$4R(p)r^2 = [R'(p)]^2, \quad 4R^2(p)\nu_2^2 = 4kR^2(p) - R_1^2(p, p)$$

At the inner points of the segments on which the region of real motions intersects the p axis, (4.5) has four solutions for $p, q, r, \nu_1, \nu_2, \nu_3$. Thus onto every inner point of the regions shown in Fig.5, four points of the integral manifold are projected. Hence into the region diffeomorphic to a ring two toruses are mapped, and into the region diffeomorphic to a rectangle one torus is mapped. Finally, in component 1 of set $\mathbf{R}^3 \setminus \Sigma$ the integral manifold consists of one two-dimensional torus, in components 2–4 it consists of two, and in component 5 it consists of four two-dimensional toruses.

We shall establish the nature of the bifurcation by analyzing the region of real motions at points Σ (whose outside contours are given in /3/) and the possible change in these regions of the number of prototypes when projected onto the pq plane of the integral surface, which in this case is not a smooth manifold (details of the method are given in /12/).

We will introduce the following notation: S is the set homeomorphic to a circle; $V = S \vee S$ is a "figure of eight"; W is the set homeomorphic to the intersection of a two-dimensional sphere with a pair of planes passing through its centre; P is the oblique product of the circle by the figure of eight (obtained from $[0, 1] \times V$ by the identification $\{0\} \times V$ with $\{1\} \times V$ with respect to a mapping that is homotopic to the central symmetry of the figure of eight); and $Q = W \times S$; $U = V \times S$.

Let γ be the boundary of the ε-neighbourhood of surface P embedded in $\mathbf{R}^3$. Then obviously $\gamma = 2T^2$. It is possible to visualize a one-parameter set of surfaces P_τ, $\tau \in (-\varepsilon, \varepsilon)$ such that $P_\tau = T^2$ when $\tau \neq 0$ and $P_0 = P$. The rearrangement $T^2 \to P \to T^2$ when τ changes we call type (1,1). Similarly we define the rearrangements of type (2,2): $2T^2 \to Q \to 2T^2$; (1,2): $T^2 \to U \to 2T^2$; (0,1): $\varnothing \to S \to T^2$. The last two rearrangements occurring in the reverse order we denote, respectively, by the symbols (2,1) and (1,0). The notation (1:1) denotes a continuous deformation of the connected component of the integral manifold on which there are no critical points. The symbols of simultaneously occurring rearrangements are connected by a plus sign, or will indicate an integral multiplier, if they are identical.

Let us now enumerate the bifurcation sequence taking place along the dash-dot arrows in Fig.1: a) $2(0,1), (2,1), (1,2), (2,2), 2(1,1)$; b) $(0,1), (1:1)+(0,1)$; c) $(0,1), (1,2)$. The transition from component 2 to component 5 from above (Fig.3) is accompanied by bifurcation $2(1:1)+2(0,1)$, and from below by $2(1,2)$. In passing from component 5 to component 3 we have bifurcation $2(2,1)$, and when emerging from component 5 into region $k<0$ we obtain bifurcation $4(1,0)$.

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ANALYTIC SOLUTIONS IN THE THEORY OF COAGULATING SYSTEMS WITH SINKS*

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Analytic solutions are derived for the problem of the evolution of the mass spectrum of three models of coagulating systems with three-dimensional uniform sinks. The case when the rate of drainage of particles with masses greater than some critical value G is higher compared with the rate of an individual act of coalescence is considered, and the problem is reduced to the consideration of a coagulation process without sinks, but where coagulation of particles of mass greater than G is forbidden. Coagulation kernels that are a) independent of the mass of the colliding particles, b) proportional to the sum, and c) equal to the product of masses of colliding particles are considered. Exact expressions are obtained for the dependence of the coagulating particles mass spectrum and for the sediment, and their asymptotic form in the limit when G is large is analyzed.

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